Emergent approximate fractal fixed-point structures in cyclic iterating image transformations

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Abstract.

Examples of an image transformation involving the shearing and folding of an image back on its own space are shown. There are invariant regions of such an area-preserving mapping, but the imperfections of computational rounding give rise to missing pixels in the iterated image. It is discovered that after a finite number of such iterations, the missing region of such an image no longer changes. It has reached a fixed point which has approximate-fractal structure. We show that when this iterating process is applied to a range of images, including a blank initial image, the result is a nearly-fixed-point image which is dominated by the fixed emergent approximate-fractal structure. When such complex systems are simulated on computers having rounding error, the application of cyclic iterating rules to a variety of natural initial systems may give rise to a final state which is more closely identifiable with the fixed point of the iteration rule (under the rounding error behaviour of the machine), than with the possibly wide range of initial states or with the true evolution of the system under study.

1. Introduction

There is a clear distinction to be made between the iteration of a rule which gives rise to an emergent complex structure in a continuous-system-variable scenario and its representation on a finite state machine such as a digital computer. In this paper we present a particularly striking example of a system which illustrates this behaviour, under the effects of the non-linear transformation compounded with the machine rounding errors.

We take an example (Ashwin-2000) from image processing. This is appropriate as the visual images can be readily appreciated by the human. We start with a square image domain. This is transformed by being stretched along a diagonal, and compressed along the other diagonal, so that the area is preserved. The diamond is then rotated through an angle which is a rational fraction of a complete rotation, such that after n rotations the image would fall back on itself. The rotated diamond is then further distorted so that the "diagonals" are not necessarily orthogonal. The resulting transformed image is mapped back onto the original square, modulo the side of the square, along x and/or y.

When this is done, it is found that the rounding errors of the computer compiler/hardware combination result in there being some missing pixels. These are set to black in the transformed image. Gradually, as the process proceeds, a fractal (Falconer) attractor unique to the combination of transformation and computer error emerges; eventually this pattern is seen whatever the starting image may have been. Of course, the images have to be precisely the same size in pixel height and width.

We suggest that this observation has important ramifications for the simulation of complex systems using computers.

2. The mapping

We start with an image, sized to lie in a square domain of N by N pixels. We consider the following transformation on the square domain.

The transformation matrix which maps (x, y) onto (X, Y) is A where

$$a = \cos(2m\pi/n)$$

with m, n integers and the elements a_{ij} of the transformation matrix A are

$$\begin{aligned} a_{11} &= \sqrt{1+a^2-b^2} - a \\ a_{12} &= b/c \\ a_{21} &= -bc \\ a_{22} &= \sqrt{1+a^2-b^2} + a \end{aligned}$$

with $|a| \leq 1$, $a^2 \leq b^2 \leq 1 + a^2$, $c \neq 0$.

You can see from this that the det $A = 1$ (so that the mapping is neither expanding nor contracting) and the modulus of its complex eigenvalues $= 1$. The modulus of the eigenvalues being equal to unity is "necessary but not sufficient" for the square to be restored by a simple modulus operation after the transformation A.

We consider a special case, which is particularly easy to visualise. In this case,

$$a = -\cos 2\pi(5/16), \quad b = 1, \quad c = 1.$$

The 5/16 of a revolution is the angle by which the image is rotated.

The special case considered here is shown in figure 1. It is easy to see how the regions of the transformed image which lie outside the original square may be moved back into the square. For convenience of computation time a square 256 pixels on a side was chosen.

We immediately notice that after 16 successive applications of this transformation, the total rotation angle is 10π, which is the first integer number of 2π radians, and the some of the original regions of the image will have mapped back onto themselves. Thus, we expect some semblance of the image to reappear every 16 iterations. This is clearly seen in figures 9 (16 iterations) and 10 (128 iterations).

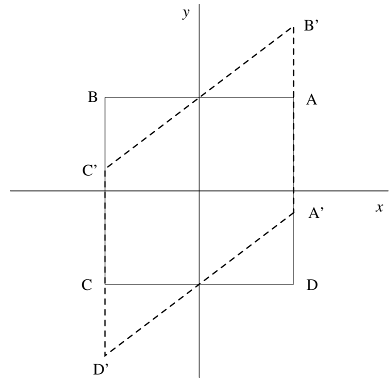

Figure 1: The specific mapping.

3. Computer experiments

In the two sequences of pictures (at the end of the paper), we show the development of two images undergoing this transformation.

The first sequence of images (figures 2-7) shows the transformation applied to a totally white blank image, resulting in black pixels where the transformation fails to return a section of the original square to a given point in the resultant image. This failure to return a point into the image is entirely a consequence of the discrete nature of the computing process, and of the rounding errors in the machine. It arguably would not happen, could we perform the process in the full continuous-variable domain. Thus, under a "continuous variable" substantiation of this process, all the pictures would consist of white squares. In this case we would need an infinite number of pixels on a side in the images, and the computing time would have extended indefinitely. However, under careful processes of definition of how limits are taken, it has been suggested (Ashwin-2000) that one might conjecture that a true fractal structure would emerge in the continuous variable limiting case.

The second sequence of images (figures 8-13), in colour in the original paper, shows a real picture undergoing the same transformation.

Looking at figure 3, we see that an ellipse consisting of sixteen missing pixels has appeared close to the centre, in the transformation of the blank image after one complete rotation, or 16 iterations. In figure 9 we discern the original image, with its broad feature preserved except near the edges, and looking carefully we may even see the structure evident in 3 beneath the distorted image.

As the sequence proceeds, more and more pixels are removed from the image. What was initially a rather small effect on the domain comes to dominate the image, until somewhere between the 1024th and 2048th iteration (figures 6 and 7) the emerging error structure reaches a fixed point and does not then develop any further. The images in 7 and 13 are markedly similar to each other; this similarity is preserved regardless of the starting image (unless it is totally black).

Looked at mathematically, the structure in figure 7 consists of all those pixels of the original image which, during the iteration process, land on one of the few pixels in each iteration which does not reappear in the succeeding substantiation of the image. This may be regarded in the same light as the missing words from the iteration sequences in our other paper (Deane).

As the iteration process proceeds beyond this fixed point for the emerging fractal structure, some of the remaining pixels may continue to move around within the image, whereas others map back onto themselves and stay fixed for ever.

4. Discussion

In many areas where computer simulations are extensively used, the effects of rounding error on the algorithms used are sometimes regarded as neglectable, or at least easily understood and readily accounted for. In this example, we have contrived an example of a computer simulation where the rounding errors give rise to a complex (approximate-fractal image) structure. If such a simulation were to be applied to a complex system for which the end result was expected to have similar approximate-fractal structure, confusion between the effects of computer rounding error and the real result of the simulation could easily arise.

The referees of this paper raised the question whose sense was "How do you know the structures you see are "true fractals"? Can you (in a sense) prove this assertion? Of course they are fixed structures as they are the end point of development on a finite state machine"

We quote from "Fractal Geometry" (Falconer). *My personal feeling is that the definition of a 'fractal' should be regarded in the same way as the biologist regards the definition of 'life'....a glance at the recent Physics literature shows the variety of natural objects that are described as fractals - cloud boundaries, topographical surfaces, coastlines, turbulence in fluids, and so on. None of these are actual fractals - their fractal features disappear if they are viewed at sufficiently small scales..... there are no true fractals in nature......* It seems reasonable to refer to such practical manifestations as "approximate-fractals".

In the case considered here, the purpose of a computer simulation of a complex system may be to predict the occurrence of such approximate-fractals. It therefore matters not that the computer simulation itself gives rise to an approximate-fractal. What this paper wishes to emphasise is, that the approximate fractal produced by the simulation of continuous-variable mathematical models, which are purported to describe the real complex situation, may give rise to completely spurious approximate-fractal structures under an iterating process on the finite-state computer, with its unavoidable rounding and truncation properties. Therefore the whole purpose of running a computer simulation to find out what will happen, is nullified.

However, the issues raised by the referees are treated in some depth in a paper (Ashwin-1997) on the oscillations in electronic digital filters caused by overflow. They find periodic "islands" in their oscillations which in our present case correspond to the invariant regions of the image, where the original image persists. They discuss at some length, using numerics and computer simulations, what the likely fractal measure of their structures are. They also discuss the effects of varying the precision of their calculations. Interested readers are referred to this paper for further information.

The initial study presented here gives rise to many intriguing questions. Among these are the following:-

- It is clear that the number of missing pixels is constant after the black fractal image has reached its fixed point. What is not so clear is what fraction of the pixels return to their initial positions after a cycle of iterations, and what fraction are itinerant, moving around in the image from one cycle to the next.

- How does the fraction f of the image which is missing, after the fixed point is reached, depend on the number N of pixels on the side of the image? Initial investigation shows that for N between 16 and 256, the fraction f remains constant, close to 0.88.

- How does the number M of iterations required to reach the fixed point depend on the number N of pixels in the side of the image? Initial investigation shows that for N up to 256, the value of M rises roughly proportional to N.

- In the case of the continuous variable limit, it is arguable that the image may be completely covered and there are no missing pixels. However, the question as to how f behaves for any discretisation cannot be settled by an appeal to the argument that f tends to zero as N tends to infinity; it may be the case that for all discrete implementations, f tends to unity as N increases. On the other hand, it has been suggested (Ashwin-2000) that it would be a reasonable *conjecture* that, if care was taken with the definitions of limit-taking, that even in the continuous limit a fractal structure might appear having about a 0.88 occupancy factor.

- Is the appearance of the emergent fractal structure dependent on the form of the rounding behaviour of the particular machine, or is it a generic property of the fact that rounding happens at all? In other words, would one see different patterns if one performed the experiment on different machines?

- How, practically, could one determine just what the transformed image (original picture) would look like, after M iterations, if the process could be performed on a true continuous-variable calculation basis?

- If one is presented with the supposed result of a computer simulation of a complex system, what tests could one carry out to determine whether or not rounding error was significantly contributing to the result?

Unlike the striking dissolution and reconstitution of the picture of Henri Poincare in the paper in Scientific American, (Crutchfield), where the reconstituted picture appears to be complete and there are no missing regions, the transformation given in this paper is in a different class, where the end point more closely represents a fixed point of the computation process than the non-linear iterated transformation under consideration.

5. Conclusions

While the fact that rounding error can be important in complex-system simulation is probably well known and widely accepted, the demonstration using images we have given in this paper brings home the point in a forceful way. It raises interesting questions about the interactions of non-linearities due to a machine implementation with the non-linearities of the complex system being simulated. It is noticeable that the experimenter may find it difficult, if not impossible, to unscramble the effects of iterated rounding in the computer from the effects (s)he is trying to uncover by the simulation.

It may indeed be true that there is an underlying true fractal for the mappings of the class discussed here; we wish to make the point that for practical purposes, we may assume that the approximate-fractal structure which emerges here is sufficiently indistinguishable from likely structures in the system being simulated, to provide a potent source of error and confusion.

References

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P Ashwin, W Chambers and G.Petkov, *Lossless digital filter overflow oscillations; approximation of invariant fractals*, Int Journal of Bifurcation and Chaos, vol 7 number 11 pages 2603-2610, (1997).

James P Crutchfield, J Doyne Farmer, Norman H Packard, and Robert S Shaw *Chaos*, Scientific American volume 255 number 6 pages 38-49 December 1986

J.H.B.Deane and D.J.Jefferies *Chaotic Dynamics and Forbidden Words* Complex Systems conference (2000) to appear.

K Falconer, *Fractal geometry: mathematical foundations and applications*, John Wiley and sons, ISBN 0-471-92287-0 (1990)

Figure 2: Blank, before iterating.

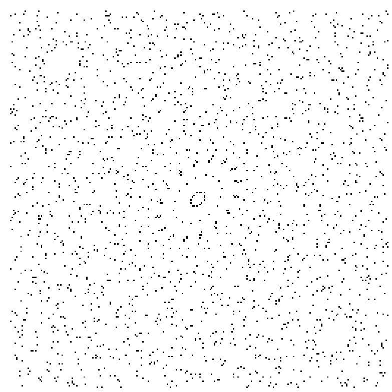

Figure 3: Blank, 16 iterations.

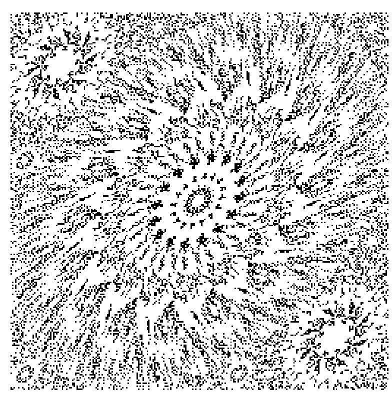

Figure 4: Blank, 128 iterations

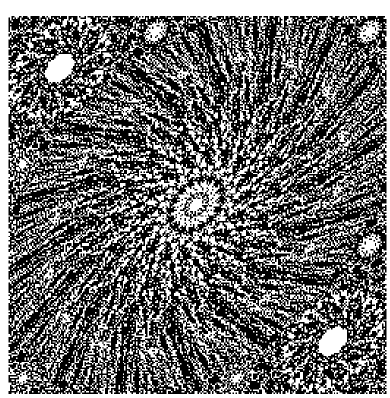

Figure 5: Blank, 512 iterations

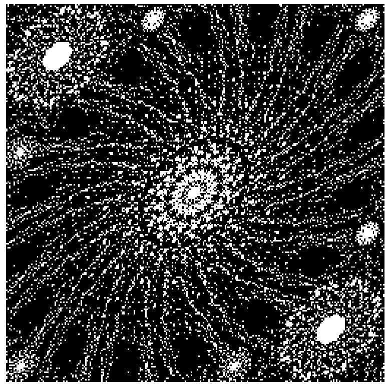

Figure 6: Blank, 1024 iterations.

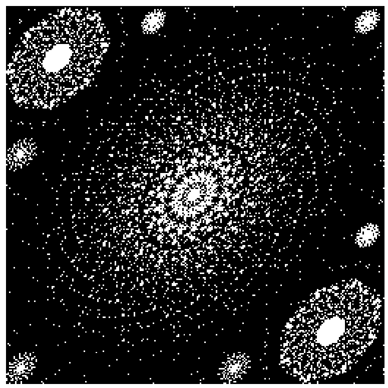

Figure 7: Blank,2048 iterations.

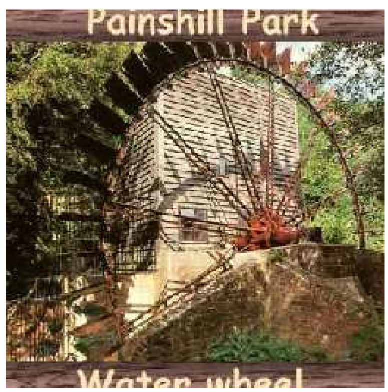

Figure 8: The original picture.

Figure 9: 16 iterations.

Figure 10: 128 iterations

Figure 11: 512 iterations

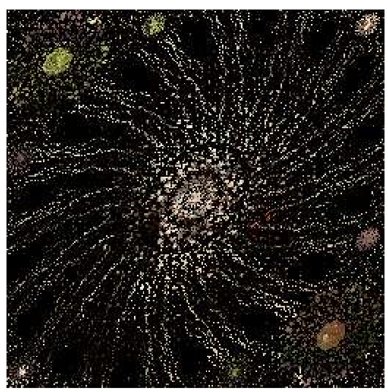

Figure 12: 1024 iterations

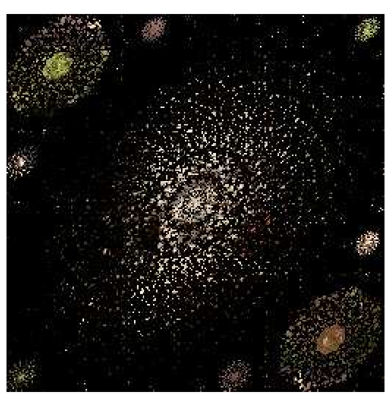

Figure 13: 2048 iterations